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Subordination results for a class of multi-term fractional Jeffreys-type equations

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Abstract

Jeffreys equation and its fractional generalizations provide extensions of the classical diffusive laws of Fourier and Fick for heat and particle transport. In this work, a class of multi-term time-fractional generalizations of the classical Jeffreys equation is studied. Restrictions on the parameters are derived, which ensure that the fundamental solution to the one-dimensional Cauchy problem is a spatial probability density function evolving in time. The studied equations are recast as Volterra integral equations with kernels represented in terms of multinomial Mittag-Leffler functions. Applying operator-theoretic approach, we establish subordination results with respect to appropriate evolution equations of integer order, depending on the considered range of parameters. Analyticity of the corresponding solution operator is also discussed. The main tools in the proofs are Laplace transform and the Bernstein functions' technique, especially, some properties of the sets of real powers of complete Bernstein functions.

Keywords Fractional calculus · Complete Bernstein function · Time-fractional Jeffreys equation · Subordination principle · Multinomial Mittag-Leffler function

Mathematics Subject Classification 26A33 · 33E12 · 35E05 · 35K05 · 35R11

1 Introduction

The classical Jeffreys equation

$$\frac{\partial}{\partial t}u(\mathbf{x},t)+a\frac{\partial^2}{\partial t^2}u(\mathbf{x},t)=\kappa\left(1+b\frac{\partial}{\partial t}\right)\Delta u(\mathbf{x},t), \tag{1.1}$$

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where $\mathbf{x} \in \mathbb{R}^n$ denotes the position vector, $t > 0$ is the temporal variable, Δ is the Laplace operator, acting with respect to the spatial variables, $\kappa > 0$, $a \geq 0$ and $b \geq 0$, generalizes both the classical heat/particle transport equation, obtained by setting $a = b = 0$ and the Cattaneo (or telegrapher's) equation, recovered by setting $b = 0$. The Jeffreys partial differential equation (1.1) can be derived from different constitutive relations with different interpretation of the coefficients: e.g. the first order differential Dual-Phase-Lag (DPL) constitutive model, introduced in [40], or the so called Jeffreys-type constitutive equation, introduced in [25] in the context of rheology. In one spatial dimension equation (1.1) is also referred to as the Guyer-Krumhansl equation, which is related to the phenomenon of second sound in solids [22]. For a concise overview of the above models and relevant references we refer to the recent work [3].

A time-fractional Jeffreys-type heat conduction equation, which generalizes (1.1), is proposed in [2, Chapter 7] as follows

$$\left(1 + aD_t^{\alpha}\right) \frac{\partial}{\partial t} u(\mathbf{x}, t) = \kappa \left(1 + bD_t^{\alpha}\right) \Delta u(\mathbf{x}, t), \tag{1.2}$$

where $a, b, \kappa > 0$, and D_t^{α} denotes the Riemann-Liouville fractional time-derivative of order $\alpha \in (0, 1)$.

Recall that the fractional derivative in the Riemann-Liouville sense of order $\delta > 0$ is defined as (see e.g. [18])

$$D_t^{\delta} f(t) = \frac{d^m}{dt^m} \int_0^t \frac{(t-\tau)^{m-\delta-1}}{\Gamma(m-\delta)} f(\tau)\, d\tau, \quad m-1 < \delta < m, \ m = 1, 2, ...$$

When δ is a positive integer, $\delta = m$, then D_t^m denotes the m-th derivative.

The one-dimensional Cauchy problem for equation (1.2) is studied analytically in [2, Chapter 7] by means of Laplace and Fourier transforms in the time and space variables, respectively, and numerical examples are given. It is established in [12] that, depending on the values of the parameter a/b, equation (1.2) governs two fundamentally different types of behavior: diffusion-like (for $a/b \leq 1$) and wave-like (for $a/b > 1$). The one-dimensional fundamental solution is shown to be a spatial probability density function evolving in time, which is unimodal in the diffusion regime and bimodal in the wave propagation regime. The multi-dimensional fundamental solutions are probability densities only in the diffusion case, while in the wave propagation case they can have negative values. In [10] equation (1.2) is studied for $a > b$ in the context of unidirectional flows of viscoelastic fluids.

A time-fractional Jeffreys-type equation in the following more general form is proposed and studied in the recent works [3, 4]:

$$\left(1 + aD_t^{\alpha}\right) \frac{\partial}{\partial t} u(\mathbf{x}, t) = \kappa D_t^{1-\gamma} \left(1 + bD_t^{\beta}\right) \Delta u(\mathbf{x}, t), \tag{1.3}$$

where $0 < \alpha, \beta, \gamma < 1$, $a, b > 0$, and $\kappa > 0$. In [4] the authors investigate the connection between the time-fractional generalization (1.3) of the Jeffreys equation and a continuous-time random walk process based on a generalized waiting time density with diverging mean. It is demonstrated that the mean squared displacement exhibits a variety of anomalous behaviors, such as retarding and accelerating subdiffusion, as well as a crossover from superdiffusion to subdiffusion. This discussion provides physics-based support for the fractional Jeffreys equation (1.3) and shows its versatility for practical applications.

Equation (1.3) in the limiting case $\gamma = 1$ is employed in the modeling of unidirectional flows of incompressible Oldroyd-B fluids, see e.g. [29, 35]. In the same context, a multi-term time-fractional equation of Jeffreys type is proposed and studied numerically in [15]. Multi-term generalizations of particular cases of equation (1.3) (e.g. with $b = 0$) are extensively studied, see e.g. the very recent works [31] and [39], where existence and uniqueness of a weak solution is established to the single-phase-lag heat equation and to a multi-term time-fractional wave equation, respectively, and properties of the solutions are deduced.

Motivated by the aforementioned developments, in the present work we study the time-fractional Jeffreys-type equation (1.3) (where for notational simplicity we set $\kappa = 1$) and the corresponding multi-term generalization

$$\left(1 + aD_t^{\alpha} + \sum_{k=1}^{K} a_k D_t^{\alpha_k}\right) \frac{\partial u}{\partial t} = D_t^{1-\gamma} \left(1 + bD_t^{\beta} + \sum_{j=1}^{J} b_j D_t^{\beta_j}\right) \Delta u \qquad (1.4)$$

with the following assumptions on the parameters

$$\begin{aligned} &0 < \alpha_k < \alpha \le 1, \;\; 0 < \beta_j < \beta \le 1, \;\; 0 < \gamma \le 1, \\ &a > 0, \;\; b > 0, \;\; a_k \ge 0, \;\; b_j \ge 0, \\ &k = 1, \ldots, K, \;\; j = 1, \ldots, J, \quad K, J \in \mathbb{N}. \end{aligned} \qquad (1.5)$$

Our main aim is to derive subordination results for this class of equations.

The concept of subordination, originally introduced by Bochner in the theory of stochastic processes, has a wide range of applications: for example in the continuous time random walk theory (see e.g. the famed paper [16] and [6, 14]), in establishing the solution of the generalized diffusion equations [37] and of the Fokker-Planck equation for the generalized geometric Brownian motion [32], or finding the solution of different diffusion processes in presence of stochastic resetting [36]. It is worth noting that the stochastic resetting is relevant in different fields of science, such as physics, chemistry, biology, economy, etc.

Recently, the subordination approach has developed into a powerful tool in the study of anomalous relaxation and diffusion processes and the physics of complex systems, see e.g. [5, 13, 19, 20, 24, 41] and the recent review paper [21]. By means of a subordination principle, it is possible to construct solutions of complex evolution equations from the solutions of classical integer order equations, or simpler fractional order ones. Subordination principle in the setting of abstract Volterra evolution equations is studied in [34], Chapter 4. Subordination results for different classes of generalized fractional diffusion-wave equations are summarized in the dissertation of the author [9].

The principle of subordination defines a hierarchy in the variety of generalized fractional evolution equations, which is essential for the proper classification and understanding of the related mathematical models. Moreover, the subordination principle is a useful tool for establishing well-posedness, for deriving integral representations of the solutions [9], for establishing regularity, asymptotic behavior, and in the study of inverse problems [33].

Subordination results concerning particular cases of equation (1.3) can be found in [10, 12], where $\alpha = \beta$, $\gamma = 1$, in [7] for $\alpha \neq \beta$, $\gamma = 1$. Subordination principle for the multi-term diffusion and diffusion-wave equations, which are particular cases of (1.4), is established in [8, 11].

The present work is devoted to the subordination principle for the class of equations (1.4). The main tools used in the proofs are Laplace transform and the Bernstein functions' technique.

The rest of the paper is organized as follows. In Section 2 definitions and basic properties of Bernstein functions and related classes of functions are listed and some properties of the sets of positive powers of complete Bernstein functions are established. In Section 3 the one-dimensional Cauchy problem for equation (1.4) is studied. In Section 4 restrictions on the parameters are found that guarantee non-negativity of the fundamental solution. Section 5 contains two general subordination theorems. Based on the results of the previous two sections, in Section 6 the subordination results concerning the class of multi-term fractional Jeffreys-type equations are formulated.

2 Complete Bernstein functions

The sets of positive integers, real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$.

The Laplace transform of a function $f(\cdot)$ is denoted as follows

$$\widehat{f}(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Definitions of four special classes of functions, related to Bernstein functions, are given next, and some of their properties, necessary for the present study, are listed. For a detailed exposition on these classes of functions we refer to [38].

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an infinitely differentiable function.

The function $\varphi(t)$ is said to be a completely monotone function ($\mathcal{CMF}$) if

$$(-1)^n \varphi^{(n)}(t) \geq 0, \quad t > 0, \quad n \in \mathbb{N}_0.$$

The characterization of the class of completely monotone functions is given by the Bernstein's theorem which states that a function is completely monotone if and only if it can be represented as the Laplace transform of a non-negative measure.

The function φ is said to be a Bernstein function ($\varphi \in \mathcal{BF}$) if

$$\varphi'(t) \in \mathcal{CMF}.$$

The class of Stieltjes functions ($\mathcal{SF}$) consists of all functions defined on $\mathbb{R}_+$ which can be represented as Laplace transform of a locally integrable completely monotone function plus a nonnegative constant. The inclusion $\mathcal{SF} \subset \mathcal{CMF}$ holds true.

The function φ is said to be a complete Bernstein functions ($\mathcal{CBF}$) if and only if

$$\varphi(s)/s \in \mathcal{SF}, \ \ s > 0.$$

There holds $\mathcal{CBF} \subset \mathcal{BF}$.

Basic examples of Stieltjes and complete Bernstein functions are the following:

$$\text{if } \ \delta \in [0, 1] \ \text{ then } \ s^{-\delta} \in \mathcal{SF}, \ \ s^{\delta} \in \mathcal{CBF}. \tag{2.1}$$

A selection of properties is listed next.

(P1) All four classes $\mathcal{CMF}$, $\mathcal{BF}$, $\mathcal{SF}$, and $\mathcal{CBF}$ are convex cones, i.e. they are closed under linear combinations with positive coefficients. Moreover, the class of completely monotone functions $\mathcal{CMF}$ is closed under pointwise multiplication.

(P2) $\mathcal{CMF} \circ \mathcal{BF} \subset \mathcal{CMF}$, where $\circ$ denotes composition of two functions.

(P3) $\varphi(s) \in \mathcal{CBF}$, $\varphi \not\equiv 0$, if and only if $1/\varphi(s) \in \mathcal{SF}$.

(P4) $\mathcal{CBF} \circ \mathcal{CBF} \subset \mathcal{CBF}$.

(P5) Let $\varphi, \psi \in \mathcal{CBF}$ and $\alpha_1, \alpha_2 \in (0, 1)$, $\alpha_1 + \alpha_2 \leq 1$. Then

$$\varphi^{\alpha_1}(s).\psi^{\alpha_2}(s) \in \mathcal{CBF}.$$

(P6) Let $\varphi, \psi \in \mathcal{CBF}$ and $\alpha \in [-1, 1]\backslash\{0\}$. Then

$$(\varphi^{\alpha}(s) + \psi^{\alpha}(s))^{1/\alpha} \in \mathcal{CBF}.$$

(P7) If $\varphi \in \mathcal{SF}$ or $\varphi \in \mathcal{CBF}$ then it can be analytically extended to the complex plane cut along the negative real axis and

$$|\arg \varphi(z)| \leq |\arg z|, \ \ z \in \mathbb{C}\backslash(-\infty, 0].$$

For proofs of the properties (P1)-(P7) we refer to [38], Chapters 6 and 7, and [34], Chapter 4.

According to (P1) the product of two completely monotone functions is again completely monotone. This, in general, does not hold for the other three classes of functions. To formulate analogous result in the context complete Bernstein functions, we introduce the following set of functions [38]

$$\mathcal{CBF}^{\delta} := \left\{\varphi^{\delta} \ : \ \varphi \in \mathcal{CBF}\right\}, \quad \delta > 0, \tag{2.2}$$

that is, $f \in \mathcal{CBF}^{\delta}$ if and only if $f^{1/\delta} \in \mathcal{CBF}$. In particular, $\mathcal{CBF}^1 \equiv \mathcal{CBF}$.

Let us first notice the inclusion

$$\mathcal{CBF}^{\delta_1} \subseteq \mathcal{CBF}^{\delta}, \quad 0 < \delta_1 \le \delta. \tag{2.3}$$

Indeed, if $f \in \mathcal{CBF}^{\delta_1}$ then $f = \varphi^{\delta_1}$ for some $\varphi \in \mathcal{CBF}$. Since $\delta_1/\delta \in (0, 1]$, property (P4) yields: $\varphi^{\delta_1/\delta} \in \mathcal{CBF}$ as a composition of two complete Bernstein functions. Therefore, $f = \left(\varphi^{\delta_1/\delta}\right)^{\delta} \in \mathcal{CBF}^{\delta}$.

In particular, (2.3) implies

$$\mathcal{CBF}^{\delta} \subseteq \mathcal{CBF}, \quad 0 < \delta \le 1. \tag{2.4}$$

Concerning the product of two functions from the above defined sets, the following result holds true.

Proposition 1 *Let* $f_1 \in \mathcal{CBF}^{\delta_1}$, $f_2 \in \mathcal{CBF}^{\delta_2}$, *where* $\delta_1 > 0, \delta_2 > 0$. *Then*

$$f_1 \cdot f_2 \in \mathcal{CBF}^{\delta_1+\delta_2}. \tag{2.5}$$

Proof We have $f_1 = \varphi^{\delta_1}$, $f_2 = \psi^{\delta_2}$, where $\varphi, \psi \in \mathcal{CBF}$. Therefore, property (P5) implies

$$(f_1 \cdot f_2)^{\frac{1}{\delta_1+\delta_2}} = \varphi^{\frac{\delta_1}{\delta_1+\delta_2}} \cdot \psi^{\frac{\delta_2}{\delta_1+\delta_2}} \in \mathcal{CBF},$$

which is equivalent to (2.5). □

Next we prove a generalization of the fact that the set $\mathcal{CBF}$ is a convex cone.

Proposition 2 *Let* $\delta \in (0, 1]$. *Then the set* $\mathcal{CBF}^{\delta}$ *is a convex cone.*

Proof Let $f_1, f_2 \in \mathcal{CBF}^{\delta}$, i.e. $f_1 = \varphi^{\delta}$ and $f_2 = \psi^{\delta}$ for some $\varphi, \psi \in \mathcal{CBF}$. Then for any $\lambda, \mu \ge 0$ property (P6) implies

$$(\lambda f_1 + \mu f_2)^{1/\delta} = \left((\lambda^{1/\delta}\varphi)^{\delta} + (\mu^{1/\delta}\psi)^{\delta}\right)^{1/\delta} \in \mathcal{CBF},$$

where we have used that the set $\mathcal{CBF}$ is closed under multiplication with a positive constant. Therefore, $\lambda f_1 + \mu f_2 \in \mathcal{CBF}^{\delta}$. □

The following multi-term result is established in [8], Proposition 3.1. For completeness, we give here a short proof in the notations of the present work.

Proposition 3 *Let* $0 \le \delta_n \le \delta \le 1$, $q_n \ge 0$, $n = 1, \ldots, N$. *Then*

$$s^{\delta} + \sum_{n=1}^{N} q_n s^{\delta_n} \in \mathcal{CBF}^{\delta} \;\; \textit{and} \;\; \left(s^{-\delta} + \sum_{n=1}^{N} q_n s^{-\delta_n}\right)^{-1} \in \mathcal{CBF}^{\delta}.$$

Proof We note first that $s^{\delta} \in \mathcal{CBF}^{\delta}$, which together with inclusion (2.3) implies $s^{\delta_n} \in \mathcal{CBF}^{\delta}$ provided $0 \le \delta_n \le \delta \le 1$. The first assertion follows then directly from Proposition 2.

For the second assertion we use property (P6) with $\alpha = -\delta$, which implies by induction that for any $f, f_n \in \mathcal{CBF}$, $n = 1, \dots, N$, there holds

$$\left(f^{-\delta} + \sum_{n=1}^{N} f_n^{-\delta}\right)^{-1} \in \mathcal{CBF}^{\delta}. \tag{2.6}$$

It remains to plug in (2.6) the complete Bernstein functions $f(s) = s$ and $f_n(s) = q_n^{-1/\delta} s^{\delta_n/\delta}$, $n = 1, \dots, N$. □

3 One-dimensional Cauchy problem

Consider the one-dimensional Cauchy problem for the fractional Jeffreys-type diffusion-wave equation (1.4), where $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$

$$\left(1 + aD_t^{\alpha} + \sum_{k=1}^{K} a_k D_t^{\alpha_k}\right) \frac{\partial u}{\partial t} = D_t^{1-\gamma}\left(1 + bD_t^{\beta} + \sum_{j=1}^{J} b_j D_t^{\beta_j}\right) \frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

with the general restrictions on the parameters (1.5) and with initial and boundary conditions

$$u(x,0) = u_0(x); \quad \lim_{t\to 0^+} \frac{\partial}{\partial t} u(x,t) = 0, \quad x \in \mathbb{R}, \tag{3.2}$$

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0. \tag{3.3}$$

Considerations leading to the second initial condition in (3.2) can be found in [12].

3.1 Fundamental solution

Let us first apply the Riemann-Liouville fractional integration operator $J_t^{1-\gamma}$ to both sides of equation (3.1), where [18]

$$J_t^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau)\, d\tau, \quad \delta > 0.$$

If $f(t)$ is a continuous function then $\lim_{t\to 0^+} (J_t^{\gamma} f)(t) = 0$, $\gamma > 0$. Therefore (see e.g. [9], eq. (1.15))

$$J_t^{1-\gamma} D_t^{1-\gamma} f(t) = f(t).$$

This implies that equation (3.1) is equivalent to

$$J_t^{1-\gamma}\left(1 + aD_t^{\alpha} + \sum_{k=1}^{K} a_k D_t^{\alpha_k}\right)\frac{\partial u}{\partial t} = \left(1 + bD_t^{\beta} + \sum_{j=1}^{J} b_j D_t^{\beta_j}\right)\frac{\partial^2 u}{\partial x^2}. \tag{3.4}$$

Problem (3.4)-(3.2)-(3.3) is treated using Laplace transform with respect to the temporal variable and Fourier transform with respect to the spatial variable.

Recall that the Fourier transform of a function $v(x)$, $x \in \mathbb{R}$, is defined by

$$\mathcal{F}\{v(x)\}(\xi) = \widetilde{v}(\xi) = \int_{-\infty}^{\infty} e^{\mathrm{i}\xi x} v(x)\, dx, \quad \xi \in \mathbb{R}.$$

The Fourier transform pair corresponding to the second order differential operator of a function $v(x)$, $x \in \mathbb{R}$, such that $\lim_{|x|\to\infty} v(x) = 0$, is

$$\mathcal{F}\{v''\}(\xi) = -|\xi|^2 \mathcal{F}\{v\}(\xi), \quad \xi \in \mathbb{R}. \tag{3.5}$$

Taking into account initial conditions (3.2), boundary condition (3.3), the formula for continuous f (see e.g. [9], eq. (1.19))

$$\mathcal{L}\{D_t^{\delta} f\}(s) = s^{\delta}\widehat{f}(s) \text{ for } \delta \in (0, 1), \tag{3.6}$$

and $\mathcal{L}\{J_t^{\delta} f\}(s) = s^{-\delta}\widehat{f}(s)$, $\delta > 0$, we derive by applying Laplace transform

$$s^{\gamma-1}\left(1 + as^{\alpha} + \sum_{k=1}^{K} a_k s^{\alpha_k}\right)(s\widehat{u} - u_0) = \left(1 + bs^{\beta} + \sum_{j=1}^{J} b_j s^{\beta_j}\right)\frac{\partial^2 \widehat{u}}{\partial x^2}.$$

Application of Fourier transform in space, yields by the use of identity (3.5) the solution in Fourier-Laplace domain

$$\widehat{\widetilde{u}}(\xi, s) = \frac{g(s)/s}{g(s) + |\xi|^2}\widetilde{u_0}(\xi), \quad \xi \in \mathbb{R}, \quad s > 0, \tag{3.7}$$

where

$$g(s) = s^{\gamma}\left(1 + as^{\alpha} + \sum_{k=1}^{K} a_k s^{\alpha_k}\right)\left(1 + bs^{\beta} + \sum_{j=1}^{J} b_j s^{\beta_j}\right)^{-1}. \tag{3.8}$$

The function $g(s)$ is characteristic for equation (3.1), as well as for the multidimensional version (1.4). It contains all information about the operators acting in time and appears in the representation as a Volterra integral equation in Section 6.

Equation (3.7) implies that the solution of the Cauchy problem (3.1)-(3.2)-(3.3) admits the integral representation

$$u(x,t) = \int_{-\infty}^{\infty} \mathcal{G}(x-y,t)u_0(y)\,dy, \quad x \in \mathbb{R}, \quad t > 0.$$

where $\mathcal{G}(x,t)$ is the fundamental solution, defined in Fourier-Laplace domain as

$$\widehat{\widehat{\mathcal{G}}}(\xi,s) = \frac{g(s)/s}{g(s)+|\xi|^2}, \quad \xi \in \mathbb{R}, \quad s > 0. \tag{3.9}$$

Inversion of the Fourier transform by using the well-known formula

$$\mathcal{F}\left\{\exp(-c|x|)\right\}(\xi) = \frac{2c}{c^2+\xi^2}, \quad c > 0; \quad x, \xi \in \mathbb{R},$$

yields the Laplace transform of the fundamental solution

$$\widehat{\mathcal{G}}(x,s) = \frac{\sqrt{g(s)}}{2s} \exp\left(-|x|\sqrt{g(s)}\right), \quad x \in \mathbb{R}. \tag{3.10}$$

Representation (3.10) implies the following property of $\mathcal{G}(x,t)$: if

$$g(s) \in \mathcal{CBF}^2 \tag{3.11}$$

then the fundamental solution $\mathcal{G}(x,t)$ is a spatial probability density function evolving in time. The proof of this property uses a standard argument. First we prove that if (3.11) is satisfied, i.e. $\sqrt{g(s)} \in \mathcal{CBF}$, then

$$\mathcal{G}(x,t) \geq 0. \tag{3.12}$$

Indeed, according to (P2), $\sqrt{g(s)} \in \mathcal{CBF} \subset \mathcal{BF}$ implies $e^{-|x|\sqrt{g(s)}} \in \mathcal{CMF}$ when $|x| > 0$ is considered as a parameter. Moreover, $\sqrt{g(s)}/s \in \mathcal{SF} \subset \mathcal{CMF}$. Then (P1) implies that $\widehat{\mathcal{G}}(x,s) \in \mathcal{CMF}$ with respect to $s > 0$ since it is a product of two completely monotone functions. Applying Bernstein's theorem yields (3.12). Further, (3.10) yields

$$\begin{aligned} \mathcal{L}\left\{\int_{-\infty}^{\infty} \mathcal{G}(x,t)dx\right\} &= \int_{-\infty}^{\infty} \widehat{\mathcal{G}}(x,s)dx \\ &= \frac{\sqrt{g(s)}}{s}\int_0^{\infty} \exp\left(-x\sqrt{g(s)}\right) dx = \frac{1}{s}. \end{aligned} \tag{3.13}$$

Applying inverse Laplace transform we obtain from (3.13)

$$\int_{-\infty}^{\infty} \mathcal{G}(x,t)dx = 1. \tag{3.14}$$

Therefore, property (3.11) for the function $g(s)$ implies that the fundamental solution $\mathcal{G}(x,t)$ is a probability density with respect to the spatial variable, which means that the considered model is physically acceptable.

In the next section we formulate sets of restrictions on the parameters, additional to the general assumptions (1.5), which guarantee that the characteristic function $g(s)$, defined in (3.8), satisfies property (3.11).

3.2 Mean squared displacement

Explicit representations of the Mean squared displacement (MSD) are derived next in terms of Mittag-Leffler functions and their multinomial generalizations and the asymptotic behavior of MSD is given.

Let us denote by $k(t) \in L^1_{loc}(\mathbb{R}_+)$ the function, which Laplace transform $\widehat{k}(s)$ satisfies the identity

$$\widehat{k}(s) = (g(s))^{-1}, \quad s > 0. \tag{3.15}$$

Representation (3.10) implies for the MSD in Laplace domain

$$\left\langle |x|^2(s) \right\rangle = \int_{\mathbb{R}} x^2 \widehat{\mathcal{G}}(x,s)\,dx = \frac{\sqrt{g(s)}}{s} \int_0^\infty x^2 \exp\left(-x\sqrt{g(s)}\right)\,dx = \frac{2}{sg(s)}.$$

where $g(s)$ is the characteristic function (3.8). Therefore

$$\left\langle |x|^2(t) \right\rangle = 2\int_0^t k(\tau)\,d\tau. \tag{3.16}$$

First, we derive expressions for the function $k(t)$ (this function will appear also later in Section 6).

In the particular case of single-term equation (1.3) with $\alpha = \beta$ and $a \neq 0$ an explicit representations for $k(t)$ can be derived in terms of Mittag-Leffler function [17]

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in \mathbb{R}, \; z \in \mathbb{C}. \tag{3.17}$$

Indeed, the relation (3.15) implies in this case

$$\widehat{k}(s) = s^{-\gamma}\left(1 + (b-a)\frac{s^\alpha}{as^\alpha + 1}\right). \tag{3.18}$$

Taking into account the Laplace transform pair [17]

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}\left(-\lambda t^\alpha\right)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}$$

we derive from (3.18) the expression

$$k(t)=\frac{t^{\gamma-1}}{\Gamma(\gamma)}+\frac{b-a}{a}t^{\gamma-1}E_{\alpha,\gamma}\left(-t^{\alpha}/a\right). \tag{3.19}$$

Using the expansion in the definition (3.17) of the Mittag-Leffler function, representation (3.19) can be rewritten as follows

$$k(t)=\frac{bt^{\gamma-1}}{a\Gamma(\gamma)}+\frac{a-b}{a^2}t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\left(-t^{\alpha}/a\right). \tag{3.20}$$

After integration and taking into account the property

$$\int_0^t \tau^{\beta-1}E_{\alpha,\beta}(-\lambda\tau^{\alpha})\,d\tau=t^{\beta}E_{\alpha,\beta+1}(-\lambda t^{\alpha}) \tag{3.21}$$

it follows

$$\begin{aligned}\left\langle |x|^2(t)\right\rangle&=\frac{2t^{\gamma}}{\Gamma(\gamma+1)}+\frac{2(b-a)}{a}t^{\gamma}E_{\alpha,\gamma+1}\left(-t^{\alpha}/a\right)\\&=\frac{2bt^{\gamma}}{a\Gamma(\gamma+1)}+\frac{2(a-b)}{a^2}t^{\alpha+\gamma}E_{\alpha,\alpha+\gamma+1}\left(-t^{\alpha}/a\right).\end{aligned} \tag{3.22}$$

Consider now the general multi-term case of function $g(s)$, given in (3.8), where $a\neq 0$. In this case

$$\widehat{k}(s)=s^{-\gamma}\left(1+as^{\alpha}+\sum_{k=1}^{K}a_k s^{\alpha_k}\right)^{-1}\left(1+bs^{\beta}+\sum_{j=1}^{J}b_j s^{\beta_j}\right) \tag{3.23}$$

and we use for the explicit representation of $k(t)$ the multinomial Mittag-Leffler function [23, 30]

$$E_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m) := \sum_{k=0}^{\infty}\sum_{\substack{k_1+\ldots+k_m=k\\ k_1\geq 0,\ldots,k_m\geq 0}}\frac{k!}{k_1!\ldots k_m!}\frac{\prod_{j=1}^{m}z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_j k_j\right)}, \tag{3.24}$$

where $z_j\in\mathbb{C}$, $\mu_j>0$, $\beta\in\mathbb{R}$, $j=1,\ldots,m$. The multinomial Mittag-Leffler function is originally introduced in [23] in the context of solution of multi-term fractional differential equations with constant coefficients, see also [8, 28, 30] for some properties. The function

$$\mathcal{E}_{\vec{\mu},\beta}(t;\vec{\lambda}) := t^{\beta-1}E_{(\mu_1,\ldots,\mu_m),\beta}(-\lambda_1 t^{\mu_1},\ldots,-\lambda_m t^{\mu_m}) \tag{3.25}$$

satisfies the Laplace transform pair [8]

$$\mathcal{L}\left\{\mathcal{E}_{\vec{\mu},\beta}(t;\vec{\lambda})\right\}(s) = s^{-\beta}\left(1+\sum_{j=1}^{m}\lambda_j s^{-\mu_j}\right)^{-1}. \tag{3.26}$$

Applying (3.26) we invert the Laplace transform in (3.23) and derive

$$\begin{aligned} k(t) = a^{-1}\mathcal{E}_{\vec{\mu},\alpha+\gamma}(t;\vec{\lambda}) + ba^{-1}\mathcal{E}_{\vec{\mu},\alpha+\gamma-\beta}(t;\vec{\lambda}) \\ + \sum_{j=1}^{J} b_j a^{-1}\mathcal{E}_{\vec{\mu},\alpha+\gamma-\beta_j}(t;\vec{\lambda}), \end{aligned} \tag{3.27}$$

where $\vec{\mu} = (\alpha, \alpha-\alpha_1, \ldots, \alpha-\alpha_K)$, $\vec{\lambda} = (1/a, a_1/a, \ldots, a_K/a)$.

Let us note that in the single-term case $a_k = b_j = 0$ expression (3.27) reduces to

$$k(t) = a^{-1}t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\left(-t^{\alpha}/a\right) + ba^{-1}t^{\gamma-1}E_{\alpha,\gamma}\left(-t^{\alpha}/a\right). \tag{3.28}$$

By applying the integration rule (3.21) to (3.28) we derive from (3.16) the representation for MSD, given in [4], Eq. (86). If $\alpha = \beta$ formula (3.28) yields (3.19) and (3.20) and the corresponding MSD, given in (3.22).

In the general case (3.27) the integration rule for multinomial functions of Mittag-Leffler type (see e.g. [8], Eq. (2.10)) yields

$$\begin{aligned} \left\langle |x|^2(t)\right\rangle = 2a^{-1}\mathcal{E}_{\vec{\mu},\alpha+\gamma+1}(t;\vec{\lambda}) + 2ba^{-1}\mathcal{E}_{\vec{\mu},\alpha+\gamma-\beta+1}(t;\vec{\lambda}) \\ + 2\sum_{j=1}^{J} b_j a^{-1}\mathcal{E}_{\vec{\mu},\alpha+\gamma-\beta_j+1}(t;\vec{\lambda}). \end{aligned} \tag{3.29}$$

By the use of the asymptotic expansions of the multinomial Mittag-Leffler functions (see [8], Eqs. (2.6) and (3.4)), we derive from (3.29)

$$\left\langle |x|^2(t)\right\rangle \sim \begin{cases} \dfrac{2ba^{-1}t^{\alpha+\gamma-\beta}}{\Gamma(\alpha+\gamma-\beta+1)}, & t \to 0^+, \\ \dfrac{2t^{\gamma}}{\Gamma(\gamma+1)}, & t \to +\infty, \end{cases} \tag{3.30}$$

which is in agreement with the asymptotic results in [4] for the single-term case.

The derived results for the MSDs (3.22) and (3.29) can be plotted using Wolfram Mathematica (the classical Mittag-Leffler functions, which appear in (3.22), are already implemented and the multinomial Mittag-Leffler functions in (3.29) can be plotted if one uses the results in Appendix E of the book [36]).

For more details on Mittag-Leffler functions we refer to [17], see also the recent book [36] and review papers [26, 27], where generalized Mittag-Leffler functions and other special functions of fractional calculus are surveyed.

4 Restrictions on the parameters

In this section sets of restrictions on the parameters are found, which guarantee that the characteristic function $g(s)$, defined in (3.8), satisfies property (3.11). More precisely, we derive inclusions of the form $g(s) \in \mathcal{CBF}^{\delta}$ for some $0 < \delta \leq 2$, which according to property (2.3) imply (3.11).

It is instructive to consider first the simpler single-term equation (1.3). In this case the function $g(s)$ admits the representation

$$g(s) = s^{\gamma}(1 + as^{\alpha})(1 + bs^{\beta})^{-1}. \tag{4.1}$$

In the limiting case $a \to 0$ it reduces to $g(s) = s^{\gamma}(1 + bs^{\beta})^{-1}$ and, therefore, $g(s) \sim b^{-1}s^{\gamma-\beta}$ as $s \to +\infty$. Thus, condition (3.11) implies $s^{\gamma-\beta} \in \mathcal{CBF}^2$, which yields

$$\beta \leq \gamma. \tag{4.2}$$

Condition (4.2) will be assumed in the next theorem. Let us note that this condition has been considered also in [4].

We are ready to formulate the first result concerning the general multi-term case.

Theorem 1 *Assume conditions (1.5) and (4.2) are satisfied. Then the function $g(s)$, defined by (3.8), obeys the property*

$$g(s) \in \mathcal{CBF}^{\alpha+\gamma}. \tag{4.3}$$

Proof Proposition 3 implies

$$1 + as^{\alpha} + \sum_{k=1}^{K} a_k s^{\alpha_k} \in \mathcal{CBF}^{\alpha},$$

$$\left(s^{-\gamma} + bs^{\beta-\gamma} + \sum_{j=1}^{J} b_j s^{\beta_j-\gamma}\right)^{-1} \in \mathcal{CBF}^{\gamma},$$

where we have used $-1 \leq -\gamma \leq \beta_j - \gamma \leq \beta - \gamma \leq 0$. Therefore, function $g(s)$, defined by (3.8), is a product of the above two functions from the classes $\mathcal{CBF}^{\alpha}$ and $\mathcal{CBF}^{\gamma}$ and Proposition 1 implies inclusion (4.3). □

Inclusion (2.4) implies directly the following

Corollary 1 *Assume conditions (1.5) are satisfied, $\beta \leq \gamma$, and $\alpha + \gamma \leq 1$. Then the characteristic function $g(s)$, defined in (3.8), is a complete Bernstein function.*

Let us note that under the conditions of Corollary 1 the function $k(t)$ in (3.23) is completely monotone as a sum of completely monotone functions. Indeed, according to Theorem 3.2 in [8] the multinomial functions of Mittag-Leffler type $\mathcal{E}_{\vec{\mu},\beta}(t; \vec{\lambda})$ is completely monotone provided $0 < \max_j\{\mu_j\} \leq \beta \leq 1$ and $\lambda_j > 0$. This is in agreement with the statement of Corollary 1, since in this case $\widehat{k}(s) \in \mathcal{SF}$, which is equivalent to $g(s) \in \mathcal{CBF}$.

Let us consider again the single-term case (4.1). A sufficient condition for $g(s) \in \mathcal{CBF}^2$ is that the function $g(s)$ is a product of two complete Bernstein functions (see Proposition 1 with $\delta_1 = \delta_2 = 1$). Thus, we seek conditions, such that $g(s) = g_1(s)g_2(s)$ with $g_1, g_2 \in \mathcal{CBF}$ and take

$$g_1(s) = s^\gamma, \quad g_2(s) = (1 + as^\alpha)(1 + bs^\beta)^{-1}.$$

Assumption $\gamma \in (0, 1]$ yields $g_1(s) \in \mathcal{CBF}$. Further, $g_2(s) \sim ab^{-1}s^{\alpha-\beta}$ as $s \to +\infty$. Therefore, $g_2(s) \in \mathcal{CBF}$ implies $\alpha \geq \beta$. On the other hand,

$$g_2(s) \sim 1 + as^\alpha - bs^\beta, \quad s \to 0.$$

Therefore, if $\alpha > \beta$, the dominant term as $s \to 0$ will be $1 - bs^\beta$, which is not even a Bernstein function (its first derivative is negative). This contradiction implies

$$\alpha = \beta. \tag{4.4}$$

This particular case of single-term equation (1.3) with (4.4) will be considered in more detail in the next theorem.

Theorem 2 *Assume the conditions (1.5) and (4.4) are satisfied. Let moreover*

$$a_k = b_j = 0 \textit{ for } k = 1, \ldots, K, \;\; j = 1, \ldots, J. \tag{4.5}$$

Then the corresponding characteristic function

$$g(s) = s^\gamma(1 + as^\alpha)(1 + bs^\alpha)^{-1}, \quad s > 0, \tag{4.6}$$

obeys the following inclusions:

(a) If $a < b$ and $\alpha \leq \gamma$ then $g(s) \in \mathcal{CBF}^\gamma$;
(b) If $a > b$ then $g(s) \in \mathcal{CBF}^{\alpha+\gamma}$.

Proof In case (a) we have the representation

$$g(s) = s^\gamma \frac{1 + as^\alpha}{1 + bs^\alpha} = \frac{a}{b}s^\gamma + (1 - a/b)\left(s^{-\gamma} + bs^{\alpha-\gamma}\right)^{-1}.$$

Since $-1 \leq -\gamma \leq \alpha - \gamma \leq 0$ then $s^\gamma \in \mathcal{CBF}^\gamma$ and $\left(s^{-\gamma} + bs^{\alpha-\gamma}\right)^{-1} \in \mathcal{CBF}^\gamma$ according to Proposition 3. Therefore, $g(s)$ is a linear combination with positive coefficients of functions from the set $\mathcal{CBF}^\gamma$. Taking into account that this set is a convex cone, see Proposition 2, we obtain $g(s) \in \mathcal{CBF}^\gamma$.

In case (b) $g(s) = s^\gamma f(s)$, where

$$f(s) = \frac{1 + as^\alpha}{1 + bs^\alpha} = 1 + \frac{a - b}{s^{-\alpha} + b}.$$

Proposition 3 yields $(s^{-\alpha} + b)^{-1} \in \mathcal{CBF}^\alpha$, which by taking into account Proposition 2 implies $f(s) \in \mathcal{CBF}^\alpha$. This together with $s^\gamma \in \mathcal{CBF}^\gamma$ shows that $g(s)$ is a product of two functions from the classes $\mathcal{CBF}^\alpha$ and $\mathcal{CBF}^\gamma$. Therefore, Proposition 1 implies $g(s) \in \mathcal{CBF}^{\alpha+\gamma}$. □

Let us extract from the last two theorems sufficient conditions for the single-term case, which guarantee $g(s) \in \mathcal{CBF}$. It is worth noting that this property (which is stronger than (3.11)) implies positivity of the fundamental solutions to multi-dimensional Cauchy problems for equation (1.3) for any dimension. Let us note that the work [4] considers only cases when the restriction $g(s) \in \mathcal{CBF}$ is satisfied.

Theorems 1 and 2 imply the following corollary for the single-term equation (1.3).

Corollary 2 *Let* $0 < \alpha, \beta, \gamma \leq 1$ *and* $a > 0, b > 0$. *The characteristic function (4.1) of the single-term equation (1.3) satisfies* $g(s) \in \mathcal{CBF}$ *provided any of the following sets of conditions is satisfied:*

(a) $\beta \leq \gamma$ *and* $\alpha + \gamma \leq 1$*;*
(b) $a < b$ *and* $\alpha = \beta \leq \gamma$*;*
(c) $a > b$*,* $\alpha = \beta$*, and* $\alpha + \gamma \leq 1$.

Proof Taking into account inclusion (2.3), conditions (a) are derived from Theorem 1, while conditions (b) and (c) are derived from Theorem 2. □

Let us note that the results of Corollary 2 expand the range of parameters, considered in [4], Proposition 1, sufficient for the property $g(s) \in \mathcal{CBF}$.

It is worth noting that the results in Corollary 2, (b) and (c), can be derived also from the relation $g(s) = (\widehat{k}(s))^{-1}$ and representations (3.19) and (3.20) for $k(t)$. It is known that the function of Mittag-Leffler type $t^{\gamma-1}E_{\alpha,\gamma}(-\lambda t^\alpha)$ is completely monotone in $t > 0$ provided $0 < \alpha \leq \gamma \leq 1$, see e.g. [17, 18]. Therefore, if $a < b$ and $\alpha \leq \gamma$ representation (3.19) yields $k(t) \in \mathcal{CMF}$. On the other hand, if $a > b$ representation (3.20) yields $k(t) \in \mathcal{CMF}$, provided $\alpha + \gamma \leq 1$. Complete monotonicity of the kernel $k(t)$ implies $\widehat{k}(s) \in \mathcal{SF}$, i.e. $g(s) \in \mathcal{CBF}$.

In order to prove a multi-term version of Theorem 2 we will need the following auxiliary result:

Proposition 4 *If* $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_N < 1$ *and* $a_n > 0, b_n > 0, \; n = 0, 1, ..., N$, *are such that*

$$\frac{a_0}{b_0} \geq \frac{a_1}{b_1} \geq \cdots \geq \frac{a_N}{b_N} \tag{4.7}$$

then

$$F(s) = \frac{\sum_{n=0}^N b_n s^{\alpha_n}}{\sum_{n=0}^N a_n s^{\alpha_n}} \in \mathcal{CBF}, \quad s > 0. \tag{4.8}$$

Proof According to the definition of complete Bernstein functions we have to prove that $F(s)/s$ is a Stieltjes function. To this end we will show that $F(s)/s$ can be represented as the Laplace transform of a completely monotone function. The inverse Laplace integral of the function $F(s)/s$, which we denote by $f(t)$, is

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} e^{st} \frac{F(s)}{s}\, ds, \quad \sigma > 0. \tag{4.9}$$

Our goal is to prove that $f(t) \in \mathcal{CMF}$. For the multivalued complex function s^δ in $F(s)$ we take the principal branch. The function under the integral sign in (4.9) has no poles in the complex plane cut along the negative real axis, since

$$\Im\left\{\sum_{n=0}^{N} a_n s^{\alpha_n}\right\} = \sum_{n=0}^{N} a_n |s|^{\alpha_n} \sin(\alpha_n \arg s) \neq 0$$

for $s \in \mathbb{C}\backslash(-\infty, 0]$. This follows from the fact that for $a_n > 0$ and $\alpha_n \in (0, 1)$ the imaginary part of any term in this sum has the same sign. We bend the contour in (4.9) into the Hankel path $Ha(\rho)$, which starts from $-\infty$ along the lower side of the negative real axis, encircles the disc $|s| = \rho$ counterclockwise and ends at $-\infty$ along the upper side of the negative real axis. The integral on the circular contour $|s| = \rho$ equals b_0/a_0 when $\rho \to 0$, since

$$\lim_{s\to 0} s \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{s\left(\sum_{n=0}^{N} a_n s^{\alpha_n}\right)} = \frac{b_0}{a_0}.$$

Therefore, taking into account the sum of the integrals along the lower and the upper sides of the negative real axis, we obtain

$$f(t) = \frac{b_0}{a_0} + \int_0^\infty e^{-rt} K(r)\, dr, \tag{4.10}$$

where

$$K(r) = -\frac{1}{\pi} \Im\left\{ \left. \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{s\left(\sum_{n=0}^{N} a_n s^{\alpha_n}\right)} \right|_{s=re^{\mathrm{i}\pi}} \right\}.$$

The expression for $K(r)$ yields

$$K(r) = \frac{1}{\pi r} \frac{\sum_{0\le i<j\le N} (a_i b_j - a_j b_i) r^{\alpha_i+\alpha_j} \sin(\alpha_j - \alpha_i)\pi}{\left(\sum_{n=0}^{N} a_n r^{\alpha_n} \cos\alpha_n\pi\right)^2 + \left(\sum_{n=0}^{N} a_n r^{\alpha_n} \sin\alpha_n\pi\right)^2}. \tag{4.11}$$

The restrictions on the parameters (4.7) imply $a_i b_j \geq a_j b_i$ and $\alpha_j > \alpha_i$ for $i < j$. Therefore, all terms in the sum in the numerator in (4.11) are non-negative and

$K(r) \geq 0$. Then representation (4.10) implies that the function $f(t)$ is completely monotone. □

Proposition 4 implies the following

Theorem 3 *Assume the conditions (1.5) and (4.4) are satisfied. Let moreover* $K = J$, $\alpha_k = \beta_k$, $a_k, b_k > 0$, $k = 1, \ldots, K$, *and*

$$0 < \alpha_1 < \cdots < \alpha_K < \alpha < 1.$$

Then the characteristic function

$$g(s) = s^\gamma \left(1 + as^\alpha + \sum_{k=1}^{K} a_k s^{\alpha_k}\right)\left(1 + bs^\alpha + \sum_{k=1}^{K} b_k s^{\alpha_k}\right)^{-1}, \quad s > 0,$$

obeys the following inclusions:

(a) If $\gamma = 1$ *and* $1 \geq a_1/b_1 \geq \cdots \geq a_K/b_K \geq a/b$ *then* $g(s) \in \mathcal{CBF}$;
(b) If $1 \geq b_1/a_1 \geq \cdots \geq b_K/a_K \geq b/a$ *then* $g(s) \in \mathcal{CBF}^{1+\gamma}$.

Proof We apply (4.8) with $N = K + 1$, $\alpha_0 = 0$, $\alpha_{K+1} = \alpha$, $a_0 = b_0 = 1$.

Case (a) follows directly from the representation $g(s) = s/F(s)$. Indeed, we proved above that $F(s)/s \in \mathcal{SF}$, which by property (P3) is equivalent to $s/F(s) \in \mathcal{CBF}$.

For case (b) we interchange the sets of parameters $\{a_n\}$ and $\{b_n\}$, $n = 0, 1, \ldots, N$, in (4.8) and in this way obtain $1/F(s) \in \mathcal{CBF}$. Then representation $g(s) = s^\gamma/F(s)$ implies $g(s) \in \mathcal{CBF}^{1+\gamma}$ by applying Proposition 1. □

Taking into account inclusion (2.3) and the general assumptions on the parameters $0 < \alpha, \beta, \gamma \leq 1$, we conclude that in all cases considered in Theorems 1-3 property (3.11) is satisfied and the considered model is physically acceptable. The established inclusions in Theorems 1-3 will be used in Section 6 to obtain subordination results for the Jeffreys-type equation with a more general spatial operator.

5 Two general subordination theorems

To formulate subordination results for the Jeffreys-type equation (1.4), based on the inclusions obtained in the previous section, we use the setting of abstract Volterra integral equations. We rewrite equation (1.4) in abstract form as equivalent Volterra integral equation and apply for the study of the obtained weaker formulation the theory developed in the monograph [34]. Next we give some definitions and general results.

Let X be a Banach space with norm $\|.\|$. Assume A is a closed linear operator in X with dense domain $D(A) \subset X$, equipped with the graph norm.

The space of all bounded linear operators from X to X is denoted by $\mathcal{B}(X)$.

Consider the abstract Volterra integral equation

$$u(t) = v + \int_0^t k(t - \tau) A u(\tau)\, d\tau, \quad t > 0, \tag{5.1}$$

with a locally integrable scalar kernel $k(t) \in L^1_{loc}(\mathbb{R}_+)$ and $v \in X$.

A function $u \in C(\mathbb{R}_+; X)$ is called a strong solution of equation (5.1) if $u \in C(\mathbb{R}_+; D(A))$ and (5.1) holds on $\mathbb{R}_+$.

Equation (5.1) is said to be well posed if for each $v \in D(A)$, there is a unique strong solution $u(t; v)$ of (5.1) and $\{v_n\} \subset D(A),\ v_n \to 0$ imply $u(t; v_n) \to 0$ in X, uniformly on compact intervals.

Suppose (5.1) is well posed. Then the solution operator $S(t)$ for (5.1) is defined by the identity

$$S(t)v = u(t; v), \quad v \in D(A),\ t \geq 0.$$

Suppose the Laplace transform $\widehat{k}(s)$ of the kernel $k(t)$ exists and $\widehat{k}(s) \neq 0$ for all $s > 0$ and set

$$g(s) = \left(\widehat{k}(s)\right)^{-1}, \quad s > 0. \tag{5.2}$$

Assume moreover that $g(s) \in \varrho(A)$ for any $s > 0$, where $\varrho(A)$ is the resolvent set of the operator A.

A solution operator $S(t)$ is called bounded if there exists a constant $C \geq 1$ such that

$$\|S(t)\| \leq C \quad \text{for all } t \geq 0.$$

Suppose $S(t)$ is a bounded solution operator for (5.1). Then the Laplace transform

$$H(s) = \int_0^\infty e^{-st} S(t)\, dt$$

is well defined for $\Re s > 0$ and is given by

$$H(s) = \frac{g(s)}{s}(g(s) - A)^{-1},$$

where the function $g(s)$ is defined in (5.2).

Consider the fractional evolution equation

$$D_t^\delta (u(t) - u(0)) = Au(t), \quad t > 0, \quad \delta \in (0, 2], \tag{5.3}$$

with initial conditions:

$$\begin{aligned} &u(0) = v \in X \text{ for } \ \delta \in (0, 1]; \\ &u(0) = v \in X,\ u'(0) = 0 \text{ for } \ \delta \in (1, 2]. \end{aligned} \tag{5.4}$$

The Cauchy problem (5.3)-(5.4) corresponds to Volterra equation (5.1) with kernel

$$k(t) = k_\delta(t) = t^{\delta-1}/\Gamma(\delta); \quad g(s) = g_\delta(s) = \left(\widehat{k_\delta}(s)\right)^{-1} = s^\delta. \tag{5.5}$$

Let us denote by $S_\delta(t)$ the solution operator to the particular Volterra integral equation with kernel (5.5) (we use the notions of well-posedness and solution operator defined above also for the equivalent fractional order Cauchy problem (5.3)-(5.4)).

Since the classical abstract Cauchy problems of first and second order are particular cases of (5.3) obtained for $\delta = 1$ and $\delta = 2$, respectively, the solution operator $S_1(t)$ is a C_0-semigroup and the solution operator $S_2(t)$ is a strongly continuous cosine family, see e.g. [1].

The following subordination theorem is proven in [9], Theorem 2.4.

Theorem 4 *Assume the Cauchy problem (5.3)-(5.4) is well posed for some δ, $0 < \delta \le 2$, and admits a bounded solution operator $S_\delta(t)$. Suppose the kernel $k(t)$ is such that $\widehat{k}(s)$ exists for $s > 0$, $\widehat{k}(s) \neq 0$ and the function $g(s) = (\widehat{k}(s))^{-1}$ satisfies the condition*

$$g(s) \in \mathcal{CBF}^{\delta}, \quad s > 0. \tag{5.6}$$

Then the Volterra integral equation (5.1) admits a bounded solution operator $S(t)$, which is related to $S_\delta(t)$ via the subordination identity

$$S(t) = \int_0^\infty \Phi(t,\tau) S_\delta(\tau)\, d\tau, \quad t > 0, \tag{5.7}$$

where $\Phi(t,\tau)$ satisfies the Laplace transform relation

$$\int_0^\infty e^{-st} \Phi(t,\tau)\, dt = \frac{g(s)^{1/\delta}}{s} \exp\left(-\tau g(s)^{1/\delta}\right).$$

Moreover, the subordination kernel $\Phi(t,\tau)$ is a unilateral probability density function (PDF) in τ when $t > 0$ is considered as a parameter, that is

$$\Phi(t,\tau) \ge 0, \quad \int_0^\infty \Phi(t,\tau)\, d\tau = 1, \qquad t, \tau > 0. \tag{5.8}$$

A solution operator $S(t)$ is said to be a bounded analytic solution operator of angle $\theta_0 \in (0, \pi/2]$ if $S(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X)$ admits an analytic extension $S(z)$ to the sector $|\arg z| < \theta_0$, which is bounded on each subsector $|\arg z| \le \theta$, where $\theta < \theta_0$.

Sufficient conditions for analyticity of the solution operator $S(t)$ are given next.

Theorem 5 *Suppose the assumptions of Theorem 4 are satisfied and there exists $\delta_0 \in (0, \delta)$, such that*

$$g(s) \in \mathcal{CBF}^{\delta_0}, \quad 0 < \delta_0 < \delta. \tag{5.9}$$

Then $S(t)$ is a bounded analytic solution operator of angle θ_*, where

$$\theta_* = \min\left\{\left(\frac{\delta}{\delta_0} - 1\right)\frac{\pi}{2}, \frac{\pi}{2}\right\}. \tag{5.10}$$

Proof According to property (P7) for complete Bernstein functions assumption (5.9) implies

$$|\arg\{g(s)^{1/\delta_0}\}| \leq |\arg s|, \quad s \in \mathbb{C}\backslash(-\infty, 0]. \tag{5.11}$$

Since $S_\delta(t)$ is a bounded solution operator, $\|S_\delta(t)\| \leq C$ for $t \geq 0$. Then the Laplace transform $H(s)$ is defined for all s, $\Re s > 0$, and satisfies

$$\|H(s)\| = \left\| \int_0^\infty e^{-st} S_\delta(t)\, dt \right\| \leq \int_0^\infty e^{-(\Re s)t} \|S_\delta(t)\|\, dt \leq \frac{C}{\Re s}, \quad \Re s > 0.$$

Moreover, $s^\delta \in \rho(A)$ for $\Re s > 0$ and $H(s) = s^{\delta-1}\left(s^\delta - A\right)^{-1}$. Therefore

$$\left\| s^{\delta-1}\left(s^\delta - A\right)^{-1} \right\| \leq \frac{C}{\Re s}, \quad \Re s > 0,$$

which implies

$$\left\| s^{\delta}\left(s^\delta - A\right)^{-1} \right\| \leq \frac{C|s|}{\Re s} = \frac{C}{\cos(\arg s)}, \quad \Re s > 0. \tag{5.12}$$

According the Generation theorem for bounded analytic solution operators ([34], Theorem 2.1.) we have to prove that for any $\theta < \theta_*$ there exists $C_1 = C_1(\theta)$, such that

$$\left\| \frac{g(s)}{s} (g(s) - A)^{-1} \right\| \leq \frac{C_1}{|s|} \tag{5.13}$$

for all $s \in \mathbb{C}\backslash\{0\}$, satisfying $|\arg s| < \theta + \pi/2$. For such s inequality (5.11) implies

$$\left| \arg g(s)^{1/\delta} \right| \leq \frac{\delta_0}{\delta}\left(\theta + \frac{\pi}{2}\right) < \frac{\delta_0}{\delta}\left(\theta_* + \frac{\pi}{2}\right) \leq \frac{\pi}{2}.$$

Thus, $\Re\left\{g(s)^{1/\delta}\right\} > 0$ and we can replace s in (5.12) by $g(s)^{1/\delta}$, which gives

$$\left\| g(s)\, (g(s) - A)^{-1} \right\| \leq \frac{C}{\cos(\arg g(s)^{1/\delta})} \leq \frac{C}{\cos\left(\frac{\delta_0}{\delta}\left(\theta + \frac{\pi}{2}\right)\right)} = C_1(\theta).$$

This implies (5.13). □

6 Subordination results for the fractional Jeffreys-type equation

Consider the Jeffreys-type evolution equation (1.4) in abstract form

$$\left(1 + aD_t^{\alpha} + \sum_{k=1}^{K} a_k D_t^{\alpha_k}\right) u'(t) = D_t^{1-\gamma} \left(1 + bD_t^{\beta} + \sum_{j=1}^{J} b_j D_t^{\beta_j}\right) Au(t), \quad (6.1)$$

where A is a closed linear densely defined operator in a Banach space X. The general assumptions on the parameters are given in (1.5). Equation (6.1) is supplemented with the initial conditions

$$u(0) = v \in X, \quad u'(0) = 0. \quad (6.2)$$

To apply the general subordination theorems from the previous section we recast the Jeffreys-type equation (6.1) as a Volterra integral equation. By applying Laplace transform and using similar argument as in Section 3, we obtain from equation (6.1) with initial conditions (6.2) the relation

$$s^{\gamma-1}\left(1 + as^{\alpha} + \sum_{k=1}^{K} a_k s^{\alpha_k}\right)(s\widehat{u}(s) - v) = \left(1 + bs^{\beta} + \sum_{j=1}^{J} b_j s^{\beta_j}\right) A\widehat{u}(s).$$

Therefore,

$$\widehat{u}(s) = v\frac{1}{s} + \frac{1 + bs^{\beta} + \sum_{j=1}^{J} b_j s^{\beta_j}}{s^{\gamma}\left(1 + as^{\alpha} + \sum_{k=1}^{K} a_k s^{\alpha_k}\right)} A\widehat{u}(s). \quad (6.3)$$

Taking the inverse Laplace transform in (6.3) yields a Volterra integral equation (5.1) with kernel $k(t)$ satisfying $\widehat{k}(s) = (g(s))^{-1}$, where $g(s)$ is the characteristic function, defined in (3.8).

Explicit representations for the kernel $k(t)$ in terms of functions of Mittag-Leffler type are already found in subsection 3.2.

Next we establish subordination results, based on Theorems 4 and 5 and the obtained in Section 4 inclusions of the form $g(s) \in \mathcal{CBF}^{\delta}$ for the characteristic function $g(s)$ and different values of the parameter δ, depending on the assumptions on the model parameters.

For example, applying Theorems 1-3, we deduce directly that Theorem 4 is satisfied for the integral equation (5.1), corresponding to the Jeffreys-type evolution equation (6.1) and the following values of δ: $\delta = \alpha + \gamma$ under the conditions of Theorem 1 and Theorem 2(b), $\delta = \gamma$ under the conditions of Theorem 2(a), $\delta = 1$ under the conditions of Theorem 3(a), and $\delta = 1 + \gamma$ under the conditions of Theorem 3(b).

The obtained results, concerning the two most important cases $\delta = 1, 2$ are formulated in the next two theorems.

Theorem 6 *Assume the operator A is a generator of a bounded strongly continuous cosine function $S_2(t)$. Suppose the coefficients of problem (6.1) satisfy the assumptions of Theorem 1, Theorem 2, or Theorem 3. Then problem (6.1) is well posed and the corresponding solution operator $S(t)$ is bounded and satisfies the following subordination relation*

$$S(t) = \int_0^\infty \Phi_1(t,\tau) S_2(\tau)\, d\tau, \quad t > 0, \tag{6.4}$$

where the kernel $\Phi_1(t,\tau)$ is defined via the Laplace transform pair

$$\int_0^\infty e^{-st} \Phi_1(t,\tau)\, dt = \frac{\sqrt{g(s)}}{s} \exp\left(-\tau\sqrt{g(s)}\right), \quad s, \tau > 0, \tag{6.5}$$

and obeys (5.8). If moreover $\gamma \neq 1$ then $S(t)$ is a bounded analytic solution operator.

As discussed in Section 4, the assumptions on the parameters in Theorem 6 ensure physical acceptability of the corresponding model. For the operator A we can take the Laplace operator Δ or a more general elliptic operator with appropriate domain, such that it generates a bounded strongly continuous cosine function (for examples see e.g. [1]). Then Theorem 6 implies that equation (1.4) is subordinated to a classical wave equation.

Theorem 7 *Let A be a generator of a bounded C_0-semigroup of operators $S_1(t)$ and suppose the coefficients of problem (6.1) satisfy the assumptions of Corollary 1, Corollary 2, or Theorem 3(a). Then problem (6.1) is well posed, the corresponding solution operator $S(t)$ is bounded and satisfies the subordination relation*

$$S(t) = \int_0^\infty \Phi_2(t,\tau) S_1(\tau)\, d\tau, \quad t > .$$

Here the kernel $\Phi_2(t,\tau)$ is defined via the Laplace transform pair

$$\int_0^\infty e^{-st} \Phi_2(t,\tau)\, dt = \frac{g(s)}{s} \exp\left(-\tau g(s)\right), \quad s, \tau > 0,$$

and obeys (5.8). In particular, under the assumptions of Corollaries 1 or 2 with $\gamma \neq 1$ the solution operator $S(t)$ is bounded analytic.

Let us note that under the conditions of Theorem 7 on the parameters (which are more restrictive than those required in Theorem 6) equation (1.4) is subordinated to the classical diffusion equation. Therefore, in this case it exhibits a diffusion-like character.

7 Concluding remarks

Subordination results are established for a Jeffreys-type evolution equations with discrete distributions of Riemann-Liouville time-derivatives, which generalize the time-fractional Jeffreys equation, recently proposed in [3, 4]. The considered equation is recast as Volterra integral equation with kernel explicitly represented in terms of multinomial Mittag-Leffler functions.

The derived subordination identities split the solution into two parts: a probability density function, which depends only on the parameters of the distribution of time-fractional derivatives and a strongly continuous C_0- semigroup or cosine family generated by the spatial operator. In the latter case the probability density function is closely related to the fundamental solution of the corresponding one-dimensional Cauchy problem. The main tools in the proofs are Laplace transform and the Bernstein functions' technique.

The obtained results generalize some existing subordination theorems concerning particular cases.

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Declarations

Conflicts of interest The author declares no conflict of interest.

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